Math 249 Lecture 19 Notes

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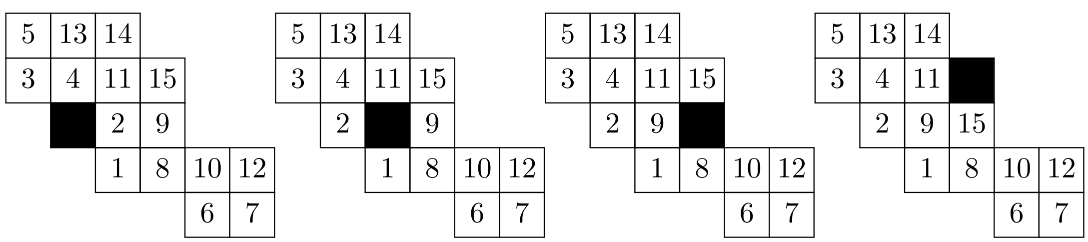
1 Jeu de Taquin Method for Littlewood-Richardson Coefficients

1.1 Jeu de taquin

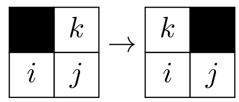
Consider the sliding tile puzzle, where you slide tiles around using an open space to place numbers in a specific order or to make a picture. We will define an analogous construction by sliding the cells of Young tableau; this is called *jeu de taquin*.

Take a standard Young tableau on a skew diagram. Pick a box on the inside that could be added to make this still a Young tableau and then play the sliding game to slide a box into the empty hole. We slide in boxes from the top or the right. To preserve the Young tableau structure, we have to fill in the box with the lesser of the two numbers to the top or right of the hole. We then have to keep sliding in boxes to fill the holes we make. This is called a *forward slide*.



We can also do a *reverse slide*, where we do the same, except we pick a box on the outside of the diagram and work our way to the inside. When we do a reverse slide, we are forced by the Young tableau structure to pick the moves that were the reverse of the moves made by the forward slide. Try looking at the above figure in reverse order! So forward and reverse slides undo each other.

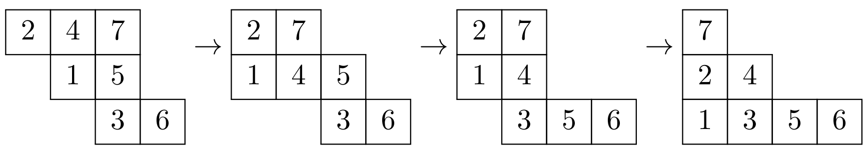
What if we want to do this for the semistandard case? We label each number that is repeated with an increasing label, like we did with RSK, and treat it like a standard Young tableau where $i_j \leq i_k$ if $j \leq k$. Slides preserve the descent set, $D(T)$. If we slide a cell down to fill the hole, then it was already less than the cells to the right on the row directly below. If we slide a cell to the left, then it is still greater than all the entries directly below and to the right. And the cell it moves over must be at least 2 less than it; otherwise, we would have



where $i = k - 1$. But then we would have $k - 1 = i < j < k$, which is impossible. So this modification to the sliding algorithm works out correctly.

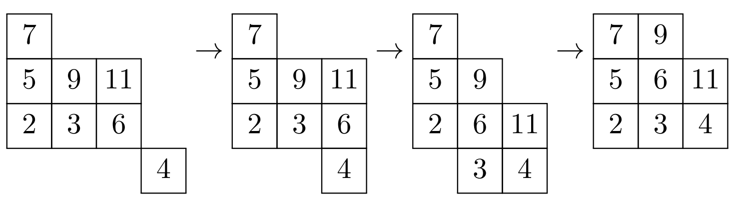
1.2 Rectification

Given a standard Young tableau on a skew diagram, we can keep performing repeated forward slides to obtain a standard Young tableau on a straight shape. This process is called *rectification*.

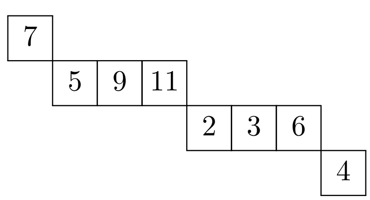


Theorem 1.1. *The rectification tableau of T, $R(T)$, is unique.*

Before we prove this, let's see some consequences. First, we should note that rectification is the same thing we do with our bumping RSK algorithm. Say we are adding 4 to a partially built tableau. We have the following picture, viewed as rectification of the tableau with the cell containing 4 adjoined:



Let's see how this works in general, by induction on the number of rows. Since the rectification produces a unique tableau, we can work with a tableau like this:



Say we have $a_1, \ldots, a_n$ and we are adding x. We will keep sliding in x until it is under a_j, where j is the largest index such that $a_j < x$; all the numbers $a_{j+2}, \ldots, a_n$ will end up to the right of x. Then we will slide down $a_1, \ldots, a_j$ (starting from a_j and working down), each time sliding a_{j+1} in its place. We end up with a tableau with the correct bottom row and with the shape of an insertion of a_{j+1} into the above rows, and induction takes care of the rest. In our example, get have a tableau like this after resolving one row:

$$\begin{array}{ccccccc} 7 & & & & & & \\ & 5 & 9 & 11 & & & \\ & & & & 6 & & \\ & & & & 2 & 3 & 4 \end{array}$$

You can then obtain the insertion tableau by rectifying the diagonal skew diagram row-wise, as we did with RSK, or column-wise.

$$\begin{array}{cccc} 4 & & & \\ & 2 & & \\ & & 1 & \\ & & & 3 \end{array} \to \begin{array}{ccc} 4 & & \\ 2 & & \\ & 3 & \\ & & 1 \end{array} \to \begin{array}{ccc} 4 & & \\ 2 & 3 & \\ & & 1 \end{array} \to \begin{array}{cc} 4 & \\ 2 & 3 \\ & 1 \end{array} \to \begin{array}{cc} 4 & \\ 2 & \\ 1 & 3 \end{array}$$

In fact, $R(T)$ is the insertion tableau for any σ that is jeu-de-taquin equivalent to T.

1.3 Relationship with Littlewood-Richardson coefficients

Theorem 1.2. *For any skew shape ν, the number c^ν_λ of Young tableau $S \in SYT(\nu)$ (or $S \in SSYT(\nu)$) such that the $R(S) = T$ depends only on ν and the shape of T.*

Corollary 1.1. *Let $s_\nu(x)$ be a skew Schur function. Then*

$$\sum_{T \in SSYT(\nu)} x^T = s_\nu(x) = \sum_{|\lambda|=n} c^\nu_\lambda s_\lambda(x),$$

and these are the Littlewood-Richardson coefficients.

Proof. By the Pieri rule, we have

$$h_k s_\lambda = \sum_{\mu/\lambda \in H_k} s_\mu.$$

So $h_{\kappa_1} \cdots h_{\kappa_\ell} s_\lambda = \sum_\mu K_{\mu,\kappa_1,\ldots,\kappa_\ell} s_\mu$, where $K_{\mu,\kappa_1,\ldots,\kappa_\ell}$ is the number of ways to μ from λ by adding horizontal strips of length $\kappa_1, \cdots, \kappa_\ell$. This is the same thing as constructing semistandard tableau of shape κ, so this coefficient is the number of SSYT of shape κ. Then

$$\langle s_\mu, h_\kappa s_\lambda \rangle = K_{\mu,\kappa_1,\ldots,\kappa_\ell} = \langle h_\kappa, s_{\mu/\lambda} \rangle.$$

This is true for all h_κ, and since any symmetric function f is a linear combination of the h_κ,

$$\langle s_\mu, f s_\lambda \rangle = \left\langle f, s_{\mu/\lambda} \right\rangle .$$

Then $c^\mu_{\kappa,\lambda} := \langle s_\mu, s_\kappa s_\lambda \rangle = \left\langle s_\kappa, s_{\mu/\lambda} \right\rangle = c^{\mu/\lambda}_\kappa$. So every Littlewood-Richardson coefficient can be thought of as a jeu de taquin coefficient. $\square$